(1) The savings function for a small country is given by

\[ S = \frac{Y^2 + 2Y + 1}{9Y + 20}, \]

where \( Y \) is annual income and \( S \) is annual savings, both measured in billions of dollars per year.

a. (5 pts) Find the nation’s marginal propensity to save, \( \frac{dS}{dY} \), and marginal propensity to consume, \( \frac{dC}{dY} \), when national income is $10 billion.

\[
\frac{dS}{dY} = \frac{(2Y + 2)(9Y + 20) - 9(Y^2 + 2Y + 1)}{(9Y + 20)^2}
\]

so

\[
\left. \frac{dS}{dY} \right|_{Y=10} = \frac{9Y^2 + 40Y + 31}{(9Y + 20)^2}
\]

and

\[
\left. \frac{dC}{dY} \right|_{Y=10} = 1 - \frac{dS}{dY}
\]

b. (3 pts) Find \( \lim_{Y \to \infty} \frac{dS}{dY} \) and interpret your answer in economic terms.

\[
\lim_{Y \to \infty} \frac{dS}{dY} = \lim_{Y \to \infty} \frac{9Y^2 + 40Y + 31}{(9Y + 20)^2} = \lim_{Y \to \infty} \frac{9Y^2 + 40Y + 31}{81Y^2 + 360Y + 400} = \lim_{Y \to \infty} \frac{9Y^2}{81Y^2} = \frac{1}{9}.
\]

**Interpretation:** When income is large, the nation will tend to save about \( \frac{1}{9} \) of every additional dollar of income.

(2) The demand equation for a monopolistic firm’s product is

\[ q = 100 - 0.25p^2, \]

where \( p \) is the price per unit (in dollars) of the firm’s product and \( q \) is the weekly demand for the firm’s product, measured in 100s of units.

a. (3 pts) Find the price-elasticity of demand for this firm’s product when the price is \( p_0 = $10 \). **Show your work.**

\[
\eta_{q/p} = \frac{dq}{dp} \cdot \frac{p}{q} = \frac{dq}{dp} \cdot \frac{p}{100 - 0.25p^2} = \frac{-0.5p^2}{100 - 0.25p^2},
\]

so

\[
\left. \eta_{q/p} \right|_{p=10} = \frac{-0.5p^2}{100 - 0.25p^2} \bigg|_{p=10} = \frac{50}{75} = \frac{2}{3}.
\]
b. (2 pts) Use your answer to a. to find the firm’s **marginal revenue** $dr/dq$ when the price is $p_0 = 10$.

$$\frac{dr}{dq}\bigg|_{p=10} = p \left( 1 + \frac{1}{\eta q/p} \right) \bigg|_{p=10} = 10 \cdot \left( 1 + \frac{1}{2/3} \right) = -5.$$ 

c. (3 pt) What will be the approximate **percentage change** in the demand for the firm’s product if they raise the price of their good from $p_0 = 10$ to $p_1 = 10.25$? **Explain your reasoning**.

**Linear approximation for percentage change:**

$$\%\Delta q \approx \eta q/p \cdot \%\Delta p = \frac{2}{3} \left( \frac{0.25}{10} \cdot 100\% \right) = \frac{2}{3} \cdot 2.5\% \approx -1.667\%$$

(3) The **marginal revenue** function for ACME Widgets Inc. is

$$\frac{dr}{dq} = \frac{10q}{0.07q^2 + 13},$$

where $q$ is monthly output, measured in 100s of widgets, and $r$ is monthly revenue, measured in $1000s.

ACME’s production function is

$$q = 5(3l + 4)^{1/2},$$

where $l$ is the number of FTEs that the firm employs per month.

An FTE is a **full-time equivalent**: a person working a 40-hour work week counts as 1 FTE, while a part-time laborer who works 30 hours a week, for example, counts as 0.75 FTEs.

a. (3 pts) Compute ACME’s **output** and **marginal product of labor**, $dq/dl$, when $l_0 = 20$.

$$q\bigg|_{l=20} = 5 \cdot 64^{1/2} = 40 \quad \text{and} \quad \frac{dq}{dl}\bigg|_{l=20} = \frac{15}{2}(3l + 4)^{1/2}\bigg|_{l=20} = \frac{15}{16}.$$ 

b. (2 pts) Compute ACME’s **marginal revenue product**, $dr/dl$, when $l_0 = 20$.

$$\frac{dr}{dl}\bigg|_{l=20} = \frac{dr}{dq}\bigg|_{q=40} \cdot \frac{dq}{dl}\bigg|_{l=20} = \frac{10q}{0.07q^2 + 13}\bigg|_{q=40} \cdot \frac{15}{16} = \frac{400}{125} \cdot \frac{15}{16} = 3.$$ 

c. (3 pts) Suppose that ACME hires a new part-time widget washer to work 10 hours a week. **By approximately how much will ACME’s **monthly revenue** change? Express your answer in dollars.**

**Linear approximation:**

$$\Delta r \approx \frac{dr}{dl}\bigg|_{l=20} \cdot \Delta l = 3 \cdot 0.25 = 0.75$$

so monthly revenue will increase by about $0.75 \cdot 1000 = 750$.

Remember — $l$ is measured in 40-hour work weeks, so an increase of 10 hours/week in labor input means that $\Delta l = 10/40 = 0.25$. 

2
Find the derivatives of the following functions. **Clean up your answers.**

a. (3 pts) \( y = \frac{3x^2 + x + 1}{5x - 2} \)

\[
\frac{dy}{dx} = \frac{(6x + 1)(5x - 2) - 5(3x^2 + x + 1)}{(5x - 2)^2} = \frac{15x^2 - 12x - 7}{(5x - 2)^2}
\]

b. (2 pts) \( g(t) = 3 \ln(t^4 + 2t - 2) \)

\[
g'(t) = 3 \cdot \frac{4t^3 + 2}{t^4 + 2t - 2} = \frac{12t^3 + 6}{t^4 + 2t - 2}
\]

c. (3 pts) \( f(u) = 5e^u \cdot u^3 \)

\[
f'(u) = 5e^u \cdot u^3 + 5e^u \cdot 3u^2 = 5e^u(u^3 + 3u^2)
\]